t-CLASS SEMIGROUPS OF NOETHERIAN DOMAINS

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ABSTRACT. The t-class semigroup of an integral domain R, denoted $\mathscr{S}_t(R)$, is the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals with the operation induced by ideal t-multiplication. This paper investigates ring-theoretic properties of a Noetherian domain that reflect reciprocally in the Clifford or Boolean property of its t-class semigroup.

1. INTRODUCTION

Let R be an integral domain. The class semigroup of R, denoted $\mathscr{S}(R)$, is the semigroup of nonzero fractional ideals modulo its subsemigroup of nonzero principal ideals [3], [19]. We define the t-class semigroup of R, denoted $\mathscr{S}_t(R)$, to be the semigroup of fractional t-ideals modulo its subsemigroup of nonzero principal ideals, that is, the semigroup of the isomorphy classes of the t-ideals of R with the operation induced by t-multiplication. Notice that $\mathscr{S}_t(R)$ stands as the t-analogue of $\mathscr{S}(R)$, whereas the class group $\mathrm{Cl}(R)$ is the t-analogue of the Picard group $\mathrm{Pic}(R)$. In general, we have

$$\mathrm{Pic}(R) \subseteq \mathrm{Cl}(R) \subseteq \mathscr{S}_t(R) \subseteq \mathscr{S}(R)$$

where the first and third containments turn into equality if R is a Prüfer domain and the second does so if R is a Krull domain.

A commutative semigroup S is said to be Clifford if every element x of S is (von Neumann) regular, i.e., there exists $a \in S$ such that $x = ax^2$. A Clifford semigroup S has the ability to stand as a disjoint union of subgroups G_e, where e ranges over the set of idempotent elements of S, and G_e is the largest subgroup of S with identity equal to e (cf. [7]). The semigroup S is said to be Boolean if for each $x \in S$, $x = x^2$. A domain R is said to be *Clifford* (resp., *Boole*) *t-regular* if $S_t(R)$ is a Clifford (resp., Boolean) semigroup.

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This paper investigates the t-class semigroups of Noetherian domains. Precisely, we study conditions under which t-stability characterizes t-regularity. Our first result, Theorem 2.2, compares Clifford t-regularity to various forms of stability. Unlike regularity, Clifford (or even Boole) t-regularity over Noetherian domains does not force the t-dimension to be one (Example 2.4). However, Noetherian strong t-stable domains happen to have t-dimension 1. Indeed, the main result, Theorem 2.6, asserts that "*R is strongly t-stable if and only if R is Boole t-regular and* $t\text{-}\dim(R) = 1$." This result is not valid for Clifford t-regularity as shown by Example 2.9. We however extend this result to the Noetherian-like larger class of strong Mori domains (Theorem 2.10).

All rings considered in this paper are integral domains. Throughout, we shall use $\mathrm{qf}(R)$ to denote the quotient field of a domain R, $\bar{I}$ to denote the isomorphy class of a t-ideal I of R in $S_t(R)$, and $\mathrm{Max}_t(R)$ to denote the set of maximal t-ideals of R.

2. MAIN RESULTS

We recall that for a nonzero fractional ideal I of R, $I_v := (I^{-1})^{-1}$, $I_t := \bigcup J_v$ where J ranges over the set of finitely generated subideals of I, and $I_w := \bigcup (I : J)$ where the union is taken over all finitely generated ideals J of R with $J^{-1} = R$. The ideal I is said to be divisorial or a v-ideal if $I = I_v$, a t-ideal if $I = I_t$, and a w-ideal if $I = I_w$. A domain R is called *strong Mori* if R satisfies the ascending chain condition on w-ideals [5]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. Suitable background on strong Mori domains is [5]. Finally, recall that the *t-dimension* of R, abbreviated $t\text{-}\dim(R)$, is by definition equal to the length of the longest chain of t-prime ideals of R.

The following lemma displays necessary and sufficient conditions for t-regularity. We often will be appealing to this lemma without explicit mention.

Lemma 2.1 ([9, Lemma 2.1])**.** *Let R be a domain.*

(1) *R is Clifford t-regular if and only if, for each t-ideal I of R, $I = (I^2(I : I^2))_t$.*
(2) *R is Boole t-regular if and only if, for each t-ideal I of R, $I = c(I^2)_t$ for some $c \neq 0 \in \mathrm{qf}(R)$.* □

An ideal I of a domain R is said to be *L-stable* (here L stands for Lipman) if $R^I := \bigcup_{n\geq 1}(I^n : I^n) = (I : I)$, and R is called L-stable if every nonzero ideal is L-stable. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to

give a characterization of Arf rings; in this context, L-stability coincides with Boole regularity [12].

Next, we state our first theorem of this section.

Theorem 2.2. *Let R be a Noetherian domain and consider the following statements:*

(1) *R is Clifford t-regular;*
(2) *Each t-ideal I of R is t-invertible in $(I:I)$;*
(3) *Each t-ideal is L-stable.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$. *Moreover, if* t-$\dim(R)=1$, *then* $(3) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$. Let I be a t-ideal of a domain A. Then for each ideal J of A, $(I:J)=(I:J_t)$. Indeed, since $J \subseteq J_t$, then $(I:J_t) \subseteq (I:J)$. Conversely, let $x \in (I:J)$. Then $xJ \subseteq I$ implies that $xJ_t=(xJ)_t \subseteq I_t=I$, as claimed. So $x \in (I:J_t)$ and therefore $(I:J) \subseteq (I:J_t)$. Now, let I be a t-ideal of R, $B=(I:I)$ and $J=I(B:I)$. Since $\overline{I}$ is regular in $\mathscr{S}_t(R)$, then $I=(I^2(I:I^2))_t=(IJ)_t$. By the claim, $B=(I:I)=(I:(IJ)_t)=(I:IJ)=((I:I):J)=(B:J)$. Since B is Noetherian, then $(I(B:I))_{t_1}=J_{t_1}=J_{v_1}=B$, where t_1- and v_1 denote the t- and v-operations with respect to B. Hence I is t-invertible as an ideal of $(I:I)$.

$(2) \Longrightarrow (3)$. Let $n \geq 1$, and $x \in (I^n:I^n)$. Then $xI^n \subseteq I^n$ implies that $xI^n(B:I) \subseteq I^n(B:I)$. So $x(I^{n-1})_{t_1}=x(I^n(B:I))_{t_1} \subseteq (I^n(B:I))_{t_1}=(I^{n-1})_{t_1}$. Now, we iterate this process by composing the two sides by $(B:I)$, applying the t-operation with respect to B and using the fact that I is t-invertible in B, we obtain that $x \in (I:I)$. Hence I is L-stable.

$(3) \Longrightarrow (1)$ Assume that t-$\dim(R)=1$. Let I be a t-ideal of R and $J=(I^2(I:I^2))_t=(I^2(I:I^2))_v$ (since R is Noetherian, and so a TV-domain). We wish to show that $I=J$. By [10, Proposition 2.8.(3)], it suffices to show that $IR_M=JR_M$ for each t-maximal ideal M of R. Let M be a t-maximal ideal of R. If $I \nsubseteq M$, then $J \nsubseteq M$. So $IR_M=JR_M=R_M$. Assume that $I \subseteq M$. Since t-$\dim(R)=1$, then $\dim(R)_M=1$. Since IR_M is L-stable, then by [12, Lemma 1.11] there exists a nonzero element x of R_M such that $I^2R_M=xIR_M$. Hence $(IR_M:I^2R_M)=(IR_M:xIR_M)=x^{-1}(IR_M:IR_M)$. So $I^2R_M(IR_M:I^2R_M)=xIR_Mx^{-1}(IR_M:IR_M)=IR_M$. Now, by [10, Lemma 5.11], $JR_M=((I^2(I:I^2))_v)R_M=(I^2(I:I^2))R_M)_v=(I^2R_M(IR_M:I^2R_M))_v=(IR_M)_v=I_vR_M=I_tR_M=IR_M$. □

According to [2, Theorem 2.1] or [8, Corollary 4.3], a Noetherian domain R is Clifford regular if and only if R is stable if and only if R is L-stable and $\dim(R)=1$. Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a Noetherian domain R to be of t-dimension one. In order to illustrate this fact, we first establish the transfer of Boole t-regularity to pullbacks issued from local Noetherian domains.

Proposition 2.3. *Let (T,M) be a local Noetherian domain with residue field K and $\phi : T \longrightarrow K$ the canonical surjection. Let k be a proper subfield of K and $R := \phi^{-1}(k)$ the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \stackrel{\phi}{\longrightarrow} & K = T/M \end{array}$$

Then R is Boole t-regular if and only if so is T.

Proof. By [4, Theorem 4] (or [6, Theorem 4.12]) R is a Noetherian local domain with maximal ideal M. Assume that R is Boole t-regular. Let J be a t-ideal of T. If $J(T:J) = T$, then $J = aT$ for some $a \in J$ (since T is local). Then $J^2 = aJ$ and so $(J^2)_{t_1} = aJ$, where t_1 is the t-operation with respect to T (note that $t_1 = v_1$ since T is Noetherian), as desired. Assume that $J(T:J) \subsetneqq T$. Since T is local with maximal ideal M, then $J(T:J) \subseteq M$. Hence $J^{-1} = (R:J) \subseteq (T:J) \subseteq (M:J) \subseteq J^{-1}$ and therefore $J^{-1} = (T:J)$. So $(T:J^2) = ((T:J):J) = ((R:J):J) = (R:J^2)$. Now, since R is Boole t-regular, then there exists $0 \neq c \in \mathrm{qf}(R)$ such that $(J^2)_t = ((J_t)^2)_t = cJ_t$. Then $(T:J^2) = (R:J^2) = (R:(J^2)_t) = (R:cJ_t) = c^{-1}(R:J_t) = c^{-1}(R:J) = c^{-1}(T:J)$. Hence $(J^2)_{t_1} = (J^2)_{v_1} = cJ_{v_1} = cJ_{t_1} = cJ$, as desired. It follows that T is Boole t-regular.

Conversely, assume that T is Boole t-regular and let I be a t-ideal of R. If $II^{-1} = R$, then $I = aR$ for some $a \in I$. So $I^2 = aI$, as desired. Assume that $II^{-1} \subsetneqq R$. Then $II^{-1} \subseteq M$. So $T \subseteq (M:M) = M^{-1} \subseteq (II^{-1})^{-1} = (I_v : I_v) = (I:I)$. Hence I is an ideal of T. If $I(T:I) = T$, then $I = aT$ for some $a \in I$ and so $I^2 = aI$, as desired. Assume that $I(T:I) \subsetneqq T$. Then $I(T:I) \subseteq M$, and so $I^{-1} \subseteq (T:I) \subseteq (M:I) \subseteq I^{-1}$. Hence $I^{-1} = (T:I)$. So $(T:I^2) = ((T:I):I) = ((R:I):I) = (R:I^2)$. But since T is Boole t-regular, then there exists $0 \neq c \in qf(T) = \mathrm{qf}(R)$ such that $(I^2)_{t_1} = ((I_{t_1})^2)_{t_1} = cI_{t_1}$. Then $(R:I^2) = (T:I^2) = (T:(I^2)_{t_1}) = (T:cI_{t_1}) = c^{-1}(T:I_{t_1}) = c^{-1}(T:I) = c^{-1}(R:I)$. Hence $(I^2)_t = (I^2)_v = cI_v = cI_t = cI$, as desired. It follows that R is Boole t-regular. □

Now we are able to build an example of a Boole t-regular Noetherian domain with t-dimension $\gneqq 1$.

Example 2.4. Let K be a field, X and Y two indeterminates over K, and k a proper subfield of K. Let $T := K[[X,Y]] = K + M$ and $R := k + M$ where $M := (X,Y)$. Since T is a UFD, then T is Boole t-regular [9, Proposition 2.2]. Further, R is a Boole t-regular Noetherian domain by Proposition 2.3. Now M is a v-ideal of R, so that t-$\dim(R) = \dim(R) = 2$.

Recall that an ideal I of a domain R is said to be *stable* (resp., *strongly stable*) if I is invertible (resp., principal) in its endomorphism ring $(I:I)$, and R is called a stable (resp., strongly stable) domain provided each nonzero ideal of R is stable (resp., strongly stable). Sally and Vasconcelos [17] used this concept to settle Bass'conjecture on one-dimensional Noetherian rings with finite integral closure. Recall that a stable domain is L-stable [1, Lemma 2.1]. For recent developments on stability, we refer the reader to [1] and [14, 15, 16]. By analogy, we define the following concepts:

Definition 2.5. A domain R is *t-stable* if each t-ideal of R is stable, and R is *strongly t-stable* if each t-ideal of R is strongly stable.

Strong t-stability is a natural stability condition that best suits Boolean t-regularity. Our next theorem is a satisfactory t-analogue for Boolean regularity [8, Theorem 4.2].

Theorem 2.6. *Let R be a Noetherian domain. The following conditions are equivalent:*

(1) *R is strongly t-stable;*
(2) *R is Boole t-regular and* $t\text{-dim}(R) = 1$.

The proof relies on the following lemmas.

Lemma 2.7. *Let R be a t-stable Noetherian domain. Then* $t\text{-dim}(R) = 1$.

Proof. Assume $t\text{-dim}(R) \geq 2$. Let $(0) \subset P_1 \subset P_2$ be a chain of t-prime ideals of R and $T := (P_2 : P_2)$. Since R is Noetherian, then so is T (as $(R:T) \neq 0$) and $T \subseteq \overline{R} = R'$, where $\overline{R}$ and R' denote respectively the complete integral closure and the integral closure of R. Let Q be any minimal prime over P_2 in T and let M be a maximal ideal of T such that $Q \subseteq M$. Then QT_M is minimal over P_2T_M which is principal by t-stability. By the principal ideal theorem, $\text{ht}(Q) = \text{ht}(QT_M) = 1$. By the Going-Up theorem, there is a height-two prime ideal Q_2 of T contracting to P_2 in R. Further, there is a minimal prime ideal Q of P_2 such that $P_2 \subseteq Q \subsetneqq Q_2$. Hence $Q \cap R = Q_2 \cap R = P_2$, which is absurd since the extension $R \subset T$ is INC. Therefore $t\text{-dim}(R) = 1$. □

Lemma 2.8. *Let R be a one-dimensional Noetherian domain. If R is Boole t-regular, then R is strongly t-stable.*

Proof. Let I be a nonzero t-ideal of R. Set $T := (I:I)$ and $J := I(T:I)$. Since R is Boole t-regular, then there is $0 \neq c \in \text{qf}(R)$ such that $(I^2)_t = cI$. Then $(T:I) = ((I:I):I) = (I:I^2) = (I:(I^2)_t) = (I:cI) = c^{-1}(I:I) = c^{-1}T$. So $J = I(T:I) = c^{-1}I$. Since J is a trace ideal of T, then $(T:J) = (J:J) = (c^{-1}I : c^{-1}I) = (I:I) = T$. Hence $J_{v_1} = T$, where v_1 is the v-operation with respect to T. Since R is one-dimensional Noetherian domain, then so is T ([11, Theorem 93]). Now, if J is a proper ideal of T, then $J \subseteq N$

for some maximal ideal N of T. Hence $T = J_{v_1} \subseteq N_{v_1} \subseteq T$ and therefore $N_{v_1} = T$. Since $\dim(T) = 1$, then each nonzero prime ideal of T is t-prime and since T is Noetherian, then $t_1 = v_1$. So $N = N_{v_1} = T$, a contradiction. Hence $J = T$ and therefore $I = cJ = cT$ is strongly t-stable, as desired. $\square$

Proof of Theorem 2.6. (1) $\Longrightarrow$ (2) Clearly R is Boole t-regular and, by Lemma 2.7, t-$\dim(R) = 1$.

(2) $\Longrightarrow$ (1) Let I be a nonzero t-ideal of R. Set $T := (I : I)$ and $J := I(T : I)$. Since R is Boole t-regular, then there is $0 \neq c \in \mathrm{qf}(R)$ such that $(I^2)_t = cI$. Then $(T : I) = ((I : I) : I) = (I : I^2) = (I : (I^2)_t) = (I : cI) = c^{-1}(I : I) = c^{-1}T$. So $J = I(T : I) = c^{-1}I$. It suffices to show that $J = T$. Since $T = (I : I) = (II^{-1})^{-1}$, then T is a divisorial (fractional) ideal of R, and since $J = c^{-1}I$, then J is a divisorial (fractional) ideal of R too. Now, for each t-maximal ideal M of R, since R_M is a one-dimensional Noetherian domain which is Boole t-regular, by Lemma 2.8, R_M is strongly t-stable. If $I \not\subseteq M$, then $T_M = (I : I)_M = (IR_M : IR_M) = R_M$ and $J_M = I(T : I)_M = R_M$. Assume that $I \subseteq M$. Then IR_M is a t-ideal of R_M. Since R_M is strongly t-stable, then $IR_M = aR_M$ for some nonzero $a \in I$. Hence $T_M = (I : I)R_M = (IR_M : IR_M) = R_M$. Then $J_M = I_M(T_M : I_M) = R_M = T_M$. Hence $J = J_t = \bigcap_{M \in \mathrm{Max}_t(R)} J_M = \bigcap_{M \in \mathrm{Max}_t(R)} T_M = T_t = T$. It follows that $I = cJ = cT$ and therefore R is strongly t-stable. $\square$

An analogue of Theorem 2.6 does not hold for Clifford t-regularity, as shown by the next example.

Example 2.9. There exists a Noetherian Clifford t-regular domain with t-$\dim(R) = 1$ such that R is not t-stable. Indeed, let us first recall that a domain R is said to be pseudo-Dedekind if every v-ideal is invertible [10]. In [18], P. Samuel gave an example of a Noetherian UFD domain R for which $R[[X]]$ is not a UFD. In [10], Kang noted that $R[[X]]$ is a Noetherian Krull domain which is not pseudo-Dedekind; otherwise, $\mathrm{Cl}(R[[X]]) = \mathrm{Cl}(R) = 0$ forces $R[[X]]$ to be a UFD, absurd. Moreover, $R[[X]]$ is a Clifford t-regular domain by [9, Proposition 2.2] and clearly $R[[X]]$ has t-dimension 1 (since Krull). But for $R[[X]]$ not being a pseudo-Dedekind domain translates into the existence of a v-ideal of $R[[X]]$ that is not invertible, as desired.

We recall that a domain R is called strong Mori if it satisfies the ascending chain condition on w-ideals. Noetherian domains are strong Mori. Next we wish to extend Theorem 2.6 to the larger class of strong Mori domains.

Theorem 2.10. *Let R be a strong Mori domain. Then the following conditions are equivalent:*

(1) *R is strongly t-stable;*
(2) *R is Boole t-regular and t-$\dim(R) = 1$.*

Proof. We recall first the following useful facts:

Fact 1 ([10, Lemma 5.11]). Let I be a finitely generated ideal of a Mori domain R and S a multiplicatively closed subset of R. Then $(I_S)_v = (I_v)_S$. In particular, if I is a t-ideal (i.e., v-ideal) of R, then I is v-finite, that is, $I = A_v$ for some finitely generated subideal A of I. Hence $(I_S)_v = ((A_v)_S)_v = ((A_S)_v)_v = (A_S)_v = (A_v)_S = I_S$ and therefore I_S is a v-ideal of R_S.

Fact 2. For each v-ideal I of R and each multiplicatively closed subset S of R, $(I : I)_S = (I_S : I_S)$. Indeed, set $I = A_v$ for some finitely generated subideal A of I and let $x \in (I_S : I_S)$. Then $xA \subseteq xA_v = xI \subseteq xI_S \subseteq I_S$. Since A is finitely generated, then there exists $\mu \in S$ such that $x\mu A \subseteq I$. So $x\mu I = x\mu A_v \subseteq I_v = I$. Hence $x\mu \in (I : I)$ and then $x \in (I : I)_S$. It follows that $(I : I)_S = (I_S : I_S)$.

$(1) \Longrightarrow (2)$ Clearly R is Boole t-regular. Let M be a maximal t-ideal of R. Then R_M is a Noetherian domain ([5, Theorem 1.9]) which is strongly t-stable. By Theorem 2.6, t-$\dim(R_M) = 1$. Since MR_M is a t-maximal ideal of R_M (Fact 1), then $\mathrm{ht}(M) = \mathrm{ht}(MR_M) = 1$. Therefore t-$\dim(R) = 1$.

$(2) \Longrightarrow (1)$ Let I be a nonzero t-ideal of R. Set $T := (I : I)$ and $J := I(T : I)$. Since R is Boole t-regular, then $(I^2)_t = cI$ for some nonzero $c \in \mathrm{qf}(R)$. So $J = c^{-1}I$. Since J and T are (fractional) t-ideals of R, to show that $J = T$, it suffices to show it t-locally. Let M be a t-maximal ideal of R. Since R_M is one-dimensional Noetherian domain which is Boole t-regular, by Theorem 2.6, R_M is strongly t-stable. By Fact 1, I_M is a t-ideal of R_M. So $I_M = a(I_M : I_M)$. Now, by Fact 2, $T_M = (I : I)_M = (I_M : I_M)$ and then $I_M = aT_M$. Hence $J_M = I_M(T_M : I_M) = T_M$, as desired. □

We close the paper with the following discussion about the limits as well as possible extensions of the above results.

Remark 2.11. (1) Unlike Clifford regularity, Clifford (or even Boole) t-regularity does not force a strong Mori domain to be Noetherian. Indeed, it suffices to consider a UFD domain which is not Noetherian.

(2) Example 2.4 provides a Noetherian Boole t-regular domain of t-dimension two. We do not know whether the assumption "t-$\dim(R) = 1$" in Theorem 2.2 can be omitted.

(3) Following [8, Proposition 2.3], the complete integral closure $\overline{R}$ of a Noetherian Boole regular domain R is a PID. We do not know if $\overline{R}$ is a UFD in the case of Boole t-regularity. However, it's the case if the conductor $(R : \overline{R}) \neq 0$. Indeed, it's clear that $\overline{R}$ is a Krull domain. But $(R : \overline{R}) \neq 0$ forces $\overline{R}$ to be Boole t-regular, when R is Boole t-regular, and by [9, Proposition 2.2], $\overline{R}$ is a UFD.

(4) The Noetherian domain provided in Example 2.4 is not strongly t-discrete since its maximal ideal is t-idempotent. We do not know if the assumption "R strongly t-discrete, i.e., R has no t-idempotent t-prime ideals" forces a Clifford t-regular Noetherian domain to be of t-dimension one.

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